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Limitations of the Carathéodory–Fejér Method for Polynomial Approximation

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Although the Carathéodory–Fejér method for obtaining polynomial approximants on a disk is quite effective for certain well-behaved functions, we show that it diverges for certain functions and, in general, does not provide better approximations than the partial sums of the Taylor expansion. © 1989 Academic Press, Inc.

1. INTRODUCTION AND RESULTS

The following theorem was proved by Carathéordory and Fejér (see, e.g., [1, p. 500]). Given a polynomial $p(z)=\sum_{k=0}^{n} c_k z^k$, there exists a unique power series extension $B(z)=p(z)+\sum_{k=n+1}^{\infty} c_k^* z^k$, analytic in the unit disk, that minimizes

$$\|B\| := \sup_{|z|<1} |B(z)|$$

among all such extensions. Moreover, $B(z)$ is a finite Blaschke product and if $p(z) \not\equiv 0$, $B(z)$ has at most n zeros.

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Here and in what follows, by a finite Blaschke product we mean a function of the form

$$\lambda \prod_{k=1}^{m} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \cdot \frac{|\alpha_k|}{\alpha_k}, \tag{1.1}$$

where $|\alpha_k| < 1$ for all k and we do not exclude the case $\lambda = 0$. If some $\alpha_k = 0$, we set the corresponding factor in (1.1) equal to $-z$.

We call $B(z)$ the *Carathéodory-Fejér* (CF) *extension* of p and sometimes we will use the notation $B_{\mathrm{CF}}(p)$ for it.

Let $f \in \mathscr{A}$, where $\mathscr{A}$ denotes the disk algebra of functions that are continuous on the closed unit disk and analytic in its interior. We equip $\mathscr{A}$ with the supremum norm $\|\cdot\|$ and let

$$E_n(f) := \inf_{p \in \Pi_n} \|f - p\|$$

denote the error of the best polynomial approximation of f by algebraic polynomials of degree at most n.

Since there are few (if any) efficient algorithms for finding best polynomial approximants on planar sets, methods that give near-optimal approximations are of particular interest. Moreover, the "goodness" of any such method should be compared to the trivial method of using the partial sums of the Taylor expansion, which gives the order of approximation $\{E_n(f) \log n\}$.

In [4], L. Trefethen proposed a method, called the Carathéodory-Fejér method, for finding polynomial approximants of functions from $\mathscr{A}$ that is based on the above minimal norm extension result. The method can be described as follows. Let $f \in \mathscr{A}$ have Taylor expansion about $z = 0$ of the form

$$f(z) \sim \sum_{k=0}^{\infty} a_k z^k.$$

The problem of best polynomial approximation to f is equivalent to the problem of minimizing

$$\left\| \sum_{k=0}^{n} c_k z^k + \sum_{k=n+1}^{\infty} a_k z^k \right\|$$

over all $(n+1)$-tuples $(c_0, ..., c_n)$. This resembles the Carathéodory-Fejér problem and the CF extensions (which are computable as the solution of certain eigenvalue problems) can be brought into the picture by using truncation and the inversion $z \to 1/z$. Thus, following Trefethen [4], we first truncate the Taylor series at some $L > n$ so that $\sum_{k=L+1}^{\infty} a_k z^k$ is negligible and we set $p(z) := \sum_{k=n+1}^{L} a_k z^k$, $p^*(z) := z^L p(1/z)$. Then we solve the CF problem for $p^*(z)$:

$$B(z) = p^*(z) + \sum_{k=L-n}^{\infty} c_k^* z^k.$$

Finally, by truncating this series again at $k = L$ and $k = L - n - 1$ and using inversion we arrive at a polynomial of degree at most n, which, when combined with the nth Taylor section for f, gives the desired approximation, which we will denote by $F_{n,L}(f; z)$ (the L indicates where we truncated the Taylor series). In terms of the a_k's and c_k^*'s we thus have

$$F_{n,L}(f; z) = \sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n} c_{L-k}^* z^k.$$

In [4] some results were obtained on the approximation properties of the CF method, but its performance for general functions has not been investigated. Despite this fact much enthusiasm has been expressed in connection with the goodness of the method and not without grounds, since in [4] it was shown that for certain well-behaved functions such as $\exp(z)$ the CF approximants are far better than the Taylor sections.

The aim of this paper is to describe the limitations of the CF method; as we will see, in general it is not better than what we can get from the Taylor sections. The results of this paper are anticipated in [2], where we found that the "near-circularity" property that the CF method was based on in [4] actually fails to hold for most of the functions in $\mathscr{A}$.

Strictly speaking the above description of the CF method is not complete since it does not state where to truncate the Taylor series or what is meant by "negligible." For certain results in [4] the truncations were performed at $L_n = 2n + 2$, but actually no fixed sequence $\{L_n\}$ can serve as universally good truncation points. In fact, we have

THEOREM 1. *If $\{L_n\}$ is an arbitrary sequence $(L_n > n)$, then there is an $f \in \mathscr{A}$ such that*

$$\limsup_{n \to \infty} F_{n, L_n}(f; 1) = \infty.$$

In fact, we can say more; namely, for most of the functions in $\mathscr{A}$ (in the sense of category), $\{F_{n, L_n}(f)\}_1^{\infty}$ fails to converge at $z = 1$ (let alone uniformly on the disk).

THEOREM 1′. *If* $\{L_n\}$ *is fixed, then the set of functions* $f \in \mathscr{A}$ *with the property*

$$\limsup_{n \to \infty} |F_{n,L_n}(f;1)| < \infty$$

is of the first category in $\mathscr{A}$.

Theorems 1 and 1′, whose proofs are deferred to Section 3, are not too surprising but suggest that one might try to improve the method by truncating the Taylor series sufficiently far depending on the function f and on n. However, as the next theorem shows, the finiteness of L is not important in the sense that *all* (sufficiently far) truncations can be uniformly bad if the resulting CF approximants are compared to best approximation.

THEOREM 2. *Suppose* $L_n > cn$, $n = 1, 2, ...,$ *for some* $c > 1$. *Then there exists an* $f \in \mathscr{A}$ *having a uniformly convergent Taylor series on* $|z| \leqslant 1$ *and a constant* $c_1 > 0$ *such that*

$$\inf_{L \geqslant L_n} |F_{n,L}(f;1) - f(1)| \geqslant c_1 E_n(f) \log n$$

holds for infinitely many n. *This* f *can be taken to be entire.*

Theorem 2, which we prove in Section 2, shows that no matter how far out ($> cn$) we truncate the Taylor expansion, we may not get a better approximation by $F_{n,L}$ than by the partial sums of the Taylor expansion. Of course, this does not contradict the fact that the CF method works well for certain subclasses of $\mathscr{A}$.

2. PROOF OF THEOREM 2

Let $S_n(f) = S_n(f, z)$ be the nth partial sum of the Taylor expansion of f about zero. (Whenever we refer to a Taylor expansion, we assume it has center at zero.)

We will need the following two simple lemmas.

LEMMA 1. *If* B *is a Blaschke product with at most* ν *zeros, then the* CF *extension of* $S_\nu(B)$ *is* B.

Proof. Let B_1 be the CF extension of $S_\nu(B)$, and set $\lambda := \|B\|$, $\lambda_1 := \|B_1\|$. If $\lambda_1 < \lambda$, then, by Rouché's theorem, $B - B_1$ has the same number of zeros in the unit disk as B, i.e., at most ν zeros. But this contradicts the fact that, since $S_\nu(B - B_1 : z) \equiv 0$, the origin is a zero of

$B - B_1$ with multiplicity at least $\nu + 1$. Thus $\lambda = \lambda_1$, and $B \equiv B_1$ follows from the uniqueness of B_1. ∎

LEMMA 2. *Let n and L be fixed. Then $F_{n,L}(f)$ is a continuous function of f. More generally, if $\mathscr{F}$ is (in $\mathscr{A}$) a compact family of functions, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $f \in \mathscr{F}$, $g \in \mathscr{A}$, and $\|f - g\| < \delta$, then $\|F_{n,L}(f) - F_{n,L}(g)\| < \varepsilon$.*

Proof. It is enough to prove the first assertion. Notice that $F_{n,L}(f; z)$ is constructed from the first $n+1$ Taylor coefficients of f and from the CF extension of a polynomial of degree $L - n - 1$ which, in turn, is formed from $L - n$ Taylor coefficients of f. Thus all we have to prove is that, for each fixed k, the kth Taylor coefficient of $B_{\mathrm{CF}}(p)$ depends continuously on p belonging to the set Π_m of polynomials of degree at most m ($m = L - n - 1$). Here $B_{\mathrm{CF}}(p)$ is understood as the CF extension of a polynomial of degree m even if some of the leading coefficients of p vanish.

It is important to notice that $B_{\mathrm{CF}}(p)$ itself is *not* a continuous function of p on Π_m. However, by Rouché's theorem, the *norm* of $B_{\mathrm{CF}}(p)$ is a continuous function of $p \in \Pi_m$ (cf. the preceding proof).

Suppose now that our claim is not true and there are a sequence $p_\nu \in \Pi_m$ and a k such that $p_\nu \to p$ as $\nu \to \infty$ in $\mathscr{A}$ and yet the kth Taylor coefficients of $B_{\mathrm{CF}}(p_\nu)$ converge to a number different from the kth Taylor coefficient of $B_{\mathrm{CF}}(p)$. Let

$$B_{\mathrm{CF}}(p_\nu) = \lambda_\nu \prod_{k=1}^{\mu_\nu} \frac{\alpha_k^{(\nu)} - z}{1 - \overline{\alpha_k^{(\nu)}} z} \frac{|\alpha_k^{(\nu)}|}{\alpha_k^{(\nu)}}, \qquad \mu_\nu \leqslant m, \ |\alpha_k^{(\nu)}| < 1,$$

and, by choosing a subsequence of $\{p_\nu\}$ if necessary, assume that all the μ_ν's are equal, say $\mu_\nu = \mu$, and the sequences $\{\lambda_\nu\}_{\nu=1}^{\infty}$ and $\{(\alpha_1^{(\nu)}, ..., \alpha_\mu^{(\nu)})\}_{\nu=1}^{\infty}$ converge to λ and $(\alpha_1, ..., \alpha_\mu)$, respectively. In case some $\alpha_j = 0$ and $\alpha_j^{(\nu)} \neq 0$ for all large ν, we can also assume that $|\alpha_j^{(\nu)}|/\alpha_j$ converges to $\exp(i\theta_j)$ and we replace λ by the product $\lambda \exp(i\theta_j)$. With this convention we set

$$B(z) := \lambda \prod_{k=1}^{\mu}{}' \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \cdot \frac{|\alpha_k|}{\alpha_k},$$

where the prime indicates that the factors with $|\alpha_k| = 1$ are omitted. Clearly, $B_{\mathrm{CF}}(p_\nu; z) \to B(z)$ as $\nu \to \infty$ uniformly on closed subsets of the interior of the unit disk; hence for every l the sequence of the lth Taylor coefficients of $B_{\mathrm{CF}}(p_\nu)$, $\nu = 1, 2, ...$, converges to the lth Taylor coefficient of B. This yields

$$S_m(B) = \lim_{\nu \to \infty} S_m(B_{\mathrm{CF}}(p_\nu)) = \lim_{\nu \to \infty} p_\nu = p,$$

while the continuity of the norm of the CF extensions implies

$$\|B\| = |\lambda| = \lim_{v \to \infty} |\lambda_v| = \lim_{v \to \infty} \|B_{CF}(p_v)\| = \|B_{CF}(p)\|.$$

Thus, B and $B_{CF}(p)$ are both minimal CF extensions of p and so $B = B_{CF}(p)$. This, however, contradicts the assumption that B and $B_{CF}(p)$ have different kth Taylor coefficients and this contradiction proves the lemma. ▮

For a given v we now construct a special Blaschke product, a suitable partial sum of which will be the basic building block for the function f of Theorem 2.

Consider the so-called Fejér polynomials

$$\sigma_v(z) := \left(\frac{1}{v} + \frac{z}{v-1} + \cdots + \frac{z^{v-1}}{1}\right) - \left(\frac{z^{v+1}}{1} + \frac{z^{v+2}}{2} + \cdots + \frac{z^{2v}}{v}\right).$$

Since for $z = e^{it}$ we have (cf. [3, 4.12.12])

$$|\sigma_v(z)| = 2\left|\sum_{k=1}^{v} \frac{\sin kt}{k}\right| < 10,$$

we get for the CF extension $B(\sigma_v) = B_{CF}(\sigma_v)$ of σ_v that

$$1 \leqslant \|B(\sigma_v)\| \leqslant 10, \tag{2.1}$$

and at the same time

$$S_v(B(\sigma_v); 1) = S_v(\sigma_v; 1) = \frac{1}{v} + \cdots + \frac{1}{1} > \log v. \tag{2.2}$$

Let the zeros of $B(\sigma_v)$ be $\alpha_1, ..., \alpha_\mu$. Suppose that of these $\alpha_1, ..., \alpha_{\mu_1}$ and only these have modulus at most $1 - v^{-8}$, and let B_v^* be the Blaschke product with $\alpha_1, ..., \alpha_{\mu_1}$ as its zeros, $B_v^*(0) > 0$, and with norm equal to $\|B(\sigma_v)\|$ (in other words, B_v^* is obtained from $B(\sigma_v)$ by dropping the Blaschke factors belonging to zeros of modulus bigger than $1 - v^{-8}$). Then B_v^* has again at most $2v$ zeros and we claim that, for large v,

$$S_v(B_v^*; 1) > \log v - 1. \tag{2.3}$$

In fact, if $\beta_k(B(\sigma_v) - B_v^*)$ denotes the kth Taylor coefficient of $B(\sigma_v) - B_v^*$, $k = 0, 1, 2, ...$, we have the upper bound

$$|\beta_k(B(\sigma_\nu)-B_\nu^*)|\leqslant\frac{10}{2\pi}\int_0^{2\pi}\left|\prod_{k=\mu_1+1}^{\mu}\frac{\alpha_k-e^{it}}{1-\overline{\alpha_k}\,e^{it}}\cdot\frac{|\alpha_k|}{\alpha_k}-1\right|dt$$
$$\leqslant\frac{10}{2\pi}\int_{[0,2\pi]\setminus\mathscr{F}}+\frac{10}{2\pi}\int_{\mathscr{F}},\tag{2.4}$$

where $\mathscr{F}$ is the union of the intervals of length $2\nu^{-4}$ having center at $\arg\alpha_k$, $\mu_1<k\leqslant\mu$ (taken mod 2π). For the integral over $[0,2\pi]\setminus\mathscr{F}$ we have the estimate

$$\frac{10}{2\pi}\int_{[0,2\pi]\setminus\mathscr{F}}\left|\prod_{k=\mu_1+1}^{\mu}\left(1-\frac{(1-|\alpha_k|^2)\,e^{it}}{\alpha_k-|\alpha_k|^2\,e^{it}}\right)|\alpha_k|-\prod_{k=\mu_1+1}^{\mu}|\alpha_k|\right|dt$$
$$+10\left|\prod_{k=\mu_1+1}^{\mu}|\alpha_k|-1\right|\leqslant\frac{C}{\nu^3},\tag{2.5}$$

where C is an absolute constant, and where we have used that for t in $[0,2\pi]\setminus\mathscr{F}$ and for every $\mu_1+1\leqslant k\leqslant\mu$,

$$\left|\frac{(1-|\alpha_k|^2)\,e^{it}}{\alpha_k-|\alpha_k|^2\,e^{it}}\right|<\frac{2(1-|\alpha_k|)}{|\alpha_k|/2\nu^4}<\frac{10}{\nu^4}$$

for ν sufficiently large.

On $\mathscr{F}$ the integrand in (2.4) is bounded by 2 and $\operatorname{meas}(\mathscr{F})\leqslant 2\nu\cdot 2\nu^{-4}=4\nu^{-3}$, which, together with (2.4) and (2.5), yields

$$|\beta_k(B(\sigma_\nu)-B_\nu^*)|\leqslant(C+80)\,\nu^{-3}.$$

Thus, we obtain the coefficients of B_ν^* from those of $B(\sigma_\nu)$ by perturbations of order at most $(C+80)\,\nu^{-3}$ and so (2.3) follows from (2.2).

Our next aim is to estimate the modulus of continuity of B_ν^* on the unit circumference. Since B_ν^* is a Blaschke product with at most 2ν zeros and of norm at most 10 (cf. (2.1)) and each of its zeros lies in the disk $\{z\colon |z|\leqslant 1-\nu^{-8}\}$, a trivial estimate yields that, for every t,

$$|(B_\nu^*)'(e^{it})|\leqslant 20\cdot 2\nu\cdot\nu^8=40\nu^9.$$

This gives that the modulus of continuity of $B_\nu^*(e^{it})$ is at most

$$\omega(B_\nu^*(e^{i\cdot});\delta)\leqslant 40\nu^9\delta,$$

and so we get for the kth partial sum of the Taylor expansion of B_ν^* the estimate (cf. [3, 5.11.7])

$$\|S_k(B_\nu^*)-B_\nu^*\|\leqslant 10^3\nu^9\,\frac{1}{k}\log k\leqslant 1$$

if $k>\nu^{10}$ and ν is sufficiently large.

Summarizing, for B_ν^* we have for all large ν

$$S_\nu(B_\nu^*; 1) > \log \nu - 1,$$

and for $k \geqslant \nu^{10}$

$$\|S_k(B_\nu^*)\| \leqslant 20.$$

Now set

$$B_\nu(z) := B_\nu^*(z)\, \frac{1/2 - z^{\nu^{10}}}{1 - z^{\nu^{10}}/2}.$$

Clearly, B_ν is a Blaschke product with at most $2\nu + \nu^{10} < 2\nu^{10}$ zeros, and since

$$\frac{1/2 - z^{\nu^{10}}}{1 - z^{\nu^{10}}/2} = \frac{1}{2} - \frac{3}{4} z^{\nu^{10}} - \frac{3}{4}\cdot\frac{1}{2} z^{2\nu^{10}} - \frac{3}{4}\cdot\frac{1}{2^2} z^{3\nu^{10}} - \cdots,$$

we have, for $k = 1, 2, \ldots$,

$$\begin{aligned} \|S_{k\nu^{10}}(B_\nu)\| &= \|\tfrac{1}{2} S_{k\nu^{10}}(B_\nu^*) - \tfrac{3}{4} z^{\nu^{10}} S_{(k-1)\nu^{10}}(B_\nu^*) - \cdots - \tfrac{3}{4}(\tfrac{1}{2})^{k-1} z^{k\nu^{10}} S_0(B_\nu^*)\| \\ &\leqslant [\tfrac{1}{2} + \tfrac{3}{4}(1 + \tfrac{1}{2} + \cdots)]\, 20 \leqslant 40 \end{aligned} \tag{2.6}$$

and at the same time (cf. (2.3))

$$\begin{aligned} &|S_{k\nu^{10}+\nu}(B_\nu; 1)| \\ &\quad = \left| \frac{1}{2} S_{k\nu^{10}+\nu}(B_\nu^*; 1) - \frac{3}{4} S_{(k-1)\nu^{10}+\nu}(B_\nu^*; 1) - \cdots - \frac{3}{4}\cdot\frac{1}{2^{k-1}} S_\nu(B_\nu^*; 1) \right| \\ &\quad > \frac{3}{4}\cdot\frac{1}{2^{k-1}} (\log \nu - 1) - \left[\frac{1}{2} + \frac{3}{4}\left(1 + \frac{1}{2} + \cdots\right)\right] 20 > \frac{1}{2^k} \log \nu - 50. \end{aligned} \tag{2.7}$$

We now return to our construction. By assumption, there is a $c > 1$ with $L_n \geqslant cn$, $n = 1, 2, \ldots$. We choose the smallest positive integer k_0 such that $(c-1)k_0 > 2$. For each ν let n_ν be defined by $n_\nu := k_0 \nu^{10} + \nu - 1$ and set

$$g_\nu(z) := S_{2\nu^{10}}\left(B_\nu; \frac{1}{z}\right) z^{2\nu^{10} + n_\nu + 1}.$$

Since the first $(n_\nu + 1)$ Taylor coefficients of g_ν vanish, in computing $F_{n_\nu, L}(g_\nu)$ for ν large and

$$L \geqslant L_{n_\nu} \geqslant cn_\nu \geqslant n_\nu + 2\nu^{10} + 1 = \deg g_\nu,$$

we have to take the CF extension of

$$g_\nu\left(\frac{1}{z}\right) z^L = S_{2\nu^{10}}(B_\nu; z)\, z^{L-2\nu^{10}-n_\nu-1}$$
$$= S_{L-n_\nu-1}(w^{L-2\nu^{10}-n_\nu-1}B_\nu(w); z).$$

Since

$$w^{L-2\nu^{10}-n_\nu-1}B_\nu(w)$$

is a Blaschke product with at most $L-n_\nu-1$ zeros, Lemma 1 gives that this CF extension coincides with

$$z^{L-2\nu^{10}-n_\nu-1}B_\nu(z).$$

Thus,

$$F_{n_\nu,L}(g_\nu; z) = -z^L\left[S_L\left(w^{L-2\nu^{10}-n_\nu-1}B_\nu(w); \frac{1}{z}\right)\right.$$
$$\left.- S_{L-n_\nu-1}\left(w^{L-2\nu^{10}-n_\nu-1}B_\nu(w); \frac{1}{z}\right)\right]$$
$$= -z^L\left(\frac{1}{z}\right)^{L-2\nu^{10}-n_\nu-1}\left[S_{2\nu^{10}+n_\nu+1}\left(B_\nu; \frac{1}{z}\right) - S_{2\nu^{10}}\left(B_\nu; \frac{1}{z}\right)\right]$$

and so (see (2.6) and (2.7)) for large ν

$$|F_{n_\nu,L}(g_\nu; 1)| \geqslant |S_{(k_0+2)\nu^{10}+\nu}(B_\nu; 1)| - |S_{2\nu^{10}}(B_\nu; 1)|$$
$$\geqslant \frac{1}{2^{k_0+2}}\log\nu - 50 - 40 \geqslant c_1 \log n_\nu,$$

where c_1 depends only on k_0, and hence on c.

What we have proved is the following: g_ν is a polynomial, $\|g_\nu\| \leqslant 40$ (cf. (2.6)), and for every $L \geqslant L_{n_\nu}$

$$|F_{n_\nu,L}(g_\nu; 1)| > c_1 \log n_\nu,$$

and so

$$|F_{n_\nu,L}(g; 1) - g(1)| > \frac{c_1}{50} E_{n_\nu}(g) \log n_\nu, \qquad L \geqslant L_{n_\nu} \tag{2.8}$$

holds for every large ν, say $\nu \geqslant \nu_0$, if $g = g_\nu$. Here c_1 depends only on c.

Choose now a sequence $\{\nu_k\}$ satisfying $n_{\nu_{k+1}} > L_m$, $m = n_{\nu_k}$, $k = 0, 1, 2, \ldots$. By Lemma 2 there exists an $\varepsilon_{k+1}^{(k)} > 0$ such that if $|b_{k+1}| < \varepsilon_{k+1}^{(k)}$, $\nu = \nu_k$, and

$g = g_{\nu_k} + b_{k+1} g_{\nu_{k+1}}$, then (2.8) holds for $L_n \leqslant L \leqslant L_m$, $n = \nu_k$, $m = \nu_{k+1}$. But then (2.8) will hold for every L since for $L > L_m$ we get from Lemma 1 and the way the CF approximants are formed that $F_{n,L}(g; z) \equiv F_{n,L_m}(g; z)$. Using again Lemma 2 we get the existance of an $\varepsilon_{k+2}^{(k)} > 0$ such that if $|b_{k+1}| < \varepsilon_{k+1}^{(k)}$, $|b_{k+2}| < \varepsilon_{k+2}^{(k)}$, $\nu = \nu_k$, and $g = g_{\nu_k} + b_{k+1} g_{\nu_{k+1}} + b_{k+2} g_{\nu_{k+2}}$, then (2.8) holds for $L_n \leqslant L \leqslant L_m$, $n = \nu_k$, $m = \nu_{k+3}$. But again then (2.8) holds for every $L_n \leqslant L$. Proceeding this way we get a sequence $\{\varepsilon_j^{(k)}\}_{j=k+1}^{\infty}$ of positive numbers such that if $|b_j| < \varepsilon_j^{(k)}$ are arbitrary, $\nu = \nu_k$, and

$$g = g_{\nu_k} + b_{k+1} g_{\nu_{k+1}} + \cdots + b_l g_{\nu_l}, \qquad l > k,$$

then (2.8) holds for g and all $L \geqslant L_n$, $n = \nu_k$. By Lemma 2,

$$|F_{n_\nu,L}(g; 1) - g(1)| \geqslant \frac{c_1}{50} E_{n_\nu}(g) \log n_\nu, \qquad \nu = \nu_k \tag{2.9}$$

holds for every $L \geqslant L_{n_\nu}$, $\nu = \nu_k$ if g is of the form

$$g = g_{\nu_k} + b_{k+1} g_{\nu_{k+1}} + \cdots$$

with $|b_j| < \varepsilon_j^{(k)}$, $j = k+1, k+2, \ldots$.

This immediately implies (2.9) for every g of the form

$$g = P + d(g_{\nu_k} + b_{k+1} g_{\nu_{k+1}} + \cdots),$$

where P is any polynomial of degree at most n_{ν_k}, and $d \neq 0$. Thus, if the sequence $\{a_k\}$ of positive numbers is sufficiently rapidly decreasing (say $a_{k+1}/a_k < \min_{1 \leqslant j \leqslant k} \varepsilon_{k+1}^{(j)}$, $k = 1, 2, \ldots$), then f defined by

$$f := \sum_{k=1}^{\infty} a_k g_{\nu_k} \tag{2.10}$$

belongs to $\mathscr{A}$ and satisfies

$$|F_{n,L}(f; 1) - f(1)| \geqslant \frac{c_1}{50} E_n(f) \log n$$

for every $n = n_{\nu_k}$ and $L \geqslant L_n$.

3. Proof of Theorems 1 and 1′

We start with the proof of Theorem 1. We distinguish two cases according to whether

$$\liminf_{n \to \infty} L_n/n = 1 \tag{3.1}$$

or not.

Case I: (3.1) holds. Let $\{n_k\}$ be a subsequence of the natural numbers such that

$$\lim_{k\to\infty} L_{n_k}/n_k = 1. \tag{3.2}$$

For each k consider the polynomials

$$P_k(z) := \sigma_{L,m,M}(z) := \left(\frac{z^M}{L-M} + \frac{z^{M+1}}{L-M-1} + \cdots + \frac{z^m}{L-m}\right)$$
$$-\left(\frac{z^{2L-m}}{L-m} + \frac{z^{2L-m+1}}{L-m+1} + \cdots + \frac{z^{2L-M}}{L-M}\right),$$

where $L = L_{n_k}$, $m = n_k$, $M = [n_k/2]$, which are the modified Fejér polynomials. For these polynomials we have (see the previous proof)

$$\|P_k\| = \|\sigma_{L,m,M}\| \leqslant 20$$

and

$$S_L(P_k; 1) = S_L(\sigma_{L,m,M}; 1) = \frac{1}{L-M} + \cdots + \frac{1}{L-m} > \log\frac{L-M}{L-m},$$

where, as before $L = L_{n_k}$, $m = n_k$, $M = [n_k/2]$. Notice that the right-hand side tends to ∞ as $k \to \infty$ because of (3.2).

The cancellation of the first $(n_k + 1)$ terms in P_k and the truncation of P_k at (the power) L_{n_k} leaves the zero polynomial and so the CF extension in forming $F_{n_k,L}(P_k)$, $L = L_{n_k}$, is identically zero. Hence

$$F_{n_k,L}(P_k) \equiv S_{n_k}(P_k),\ L = L_{n_k},$$

which means that, for any $a_k > 0$,

$$F_{n_k,L}(a_k P_k; 1) > a_k \log\frac{L-[n_k/2]}{L-n_k}, \qquad L = L_{n_k} \tag{3.3}$$

and here the right-hand-side tends to ∞ if $a_k > 0$ tends to zero sufficiently slowly. By selecting a subsequence of $\{n_k\}$ if necessary, we may assume that $\{n_k\}$ is so sparse that $n_{k+1} > 6n_k$, $3n_k > L_{n_k}$ are true and that

$$\frac{1}{k^2}\log\frac{L-[n_k/2]}{L-n_k} \to \infty, \qquad L = L_{n_k}, \tag{3.4}$$

as $k \to \infty$.

By setting

$$f(z) := \sum_{k=1}^{\infty} \frac{1}{k^2} P_k(z)$$

we get a function $f \in \mathscr{A}$ such that for $n = n_k$, $L = L_{n_k}$, $k = 1, 2, \ldots$,

$$F_{n,L}(f) \equiv F_{n,L}\left(\frac{1}{k^2} P_k\right) + \sum_{j=1}^{k-1} \frac{1}{j^2} P_j,$$

and so by setting $a_k = 1/k^2$ in (3.3) we obtain from (3.4) that

$$\lim_{k \to \infty} F_{n,L}(f; 1) = \infty, \qquad n = n_k, L = L_{n_k}.$$

This proves the result for the case of (3.1).

Case II. Suppose now that

$$\liminf_{n \to \infty} L_n/n > 1,$$

i.e., there is a $c > 1$ such that $L_n > cn$. In this case we can utilize the construction of the proof of Theorem 2 and for a suitable f of the form (2.10) with $a_k = 1/k^2$ and sufficiently rapidly increasing ν_k (say $\nu_{k+1} > L_{\nu_k}$, $\nu_k > \exp(\exp k)$, $k = 1, 2, \ldots$) we get again

$$\limsup_{n \to \infty} F_{n,L_n}(f, 1) = \infty.$$

The proof of Theorem 1′ is a simple category argument. In fact, set

$$S_N := \{f \in \mathscr{A} : |F_{n,L_n}(f; 1)| \leqslant N \text{ for all } n\}.$$

By Lemma 2 each S_N is closed in $\mathscr{A}$. Thus, if the statement of Theorem 1′ were false, then some S_N would contain a ball. But then it would contain a function of the form

$$g = P + cf,$$

where P is a polynomial, $c > 0$, and f is the function from Theorem 1. Since for $n > \deg P$

$$F_{n,L_n}(g) - g = c(F_{n,L_n}(f) - f),$$

this is impossible and the contradiction obtained proves Theorem 1′.

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